Flower-Snarks are Flow-Critical

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\textbf{Abstract}

In this paper we show that all flower-snarks are 4-flow-critical graphs.

\textit{Keywords:} nowhere-zero \(k\)-flows, Tutte’s Conjectures, flower-snarks, 3-edge-colouring

\section{Snarks and Flower-Snarks}

A \textit{snark} is a cubic graph that does not admit a 3-edge-colouring, which is cyclically 4-edge-connected and has girth at least five. The Petersen graph

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was the first snark discovered and, for several decades later, very few other
snarks were found. This apparent difficulty in finding such graphs inspired
Descartes to name them snarks after the Lewis Carroll poem "The Hunting of
the Snark". Good accounts on the delightful history behind snarks hunt and
their relevance in Graph Theory can be found in [2,7,1]. In 1975, Isaacs [5]
showed that there were infinite families of snarks, one of such being the flower-
snarks.

A flower-snark $J_n$ is a graph on $4n$ vertices, for $n \geq 5$ and odd, whose
vertices are labelled $V_i = \{w_i, x_i, y_i, z_i\}$, for $1 \leq i \leq n$ and whose edges can
be partitioned into $n$ star graphs and two cycles as we will next describe. Each
quadruple $V_i$ of vertices induce a star graph with $w_i$ as its center. Vertices
$z_i$ induce an odd cycle $(z_1, z_2, \ldots, z_n, z_1)$. Vertices $x_i$ and $y_i$ induce an
even cycle $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, x_1)$. Figure 1 depicts several different
drawings of flower-snark $J_5$ on 20 vertices. The drawing of Figure 1(a) is the
original drawing presented by Isaacs to represent flower-snarks. It emphasizes
the flower-shape of such graphs as well as the crossing of edges $(x_1, y_n)$ and
$(x_n, y_1)$. A more symmetric drawing of flower-snarks can be obtained from
the original drawing by switching the positions of vertices $x_i$ and $y_i$ for even
$i$. Figure 1(b) depicts the symmetric drawing of $J_5$ thus obtained. We present
a third representation of flower-snarks as a sequential string of stars induced
by vertices $V_i$, each of which linked to $V_{i+1}$ by a set of three linking edges
$E_i = \{(z_i, z_{i+1}), (x_i, x_{i+1}), (y_i, y_{i+1})\}$, for $1 \leq i \leq n - 1$ and closed by linking
edges $E_n = \{(z_1, z_n), (x_1, y_n), (x_n, y_1)\}$. Figure 1(c) depicts a string drawing
of $J_5$.

The symmetric drawing of a flower-snark is helpful in identifying its auto-
morphisms. Clearly, there is an automorphism mapping every $x_i$ vertex into
$y_i$ and vice-versa. Also, these graphs have rotational symmetry, i. e., an au-
tomorphism mapping every $z_i$ into $z_{i+1}$, $w_i$ into $w_{i+1}$, $x_i$ into $y_{i+1}$ and $y_i$ into
$x_{i+1}$\textsuperscript{5}. An appropriate combination of these two automorphisms can be used
to map every vertex $z_i$ into $z_j$, every vertex $w_i$ into $w_j$ and every vertex $x_i$
or $y_i$ into $x_j$ or $y_j$. Therefore, there are essentially just four different types of
edges in any flower-snark, which are (i) edges $z_i z_{i+1}$, which are called internal
edges; (ii) edges $z_i w_i$, which are called spoke edges; (iii) edges $w_i x_i$ and $w_i y_i$, which
are called fork edges; (iv) edges $x_i x_{i+1}$, $y_i y_{i+1}$, $x_1 y_n$ and $x_n y_1$, which
are called external edges.

\textsuperscript{5} These sums should be taken modulo $n$
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Let $k > 1$ be an integer, $G = (V, E)$ be a graph and $(D, \varphi)$ a pair of functions such that $D$ defines orientations to the edges of $E$ and $\varphi$ associates to each edge in $E$ a non-zero integer weight in the set $\{1, 2, \ldots, k - 1\}$. The pair $(D, \varphi)$ is a (nowhere-zero) $k$-flow of $G$ if, for every vertex $v \in V$, the sum of the weights of all edges leaving $v$ equal the sum of all edges entering $v$.

Tutte [6] defined the concept of $k$-flows with the purpose of generalizing to non-planar graphs several problems regarding colourings of planar graphs. A famous theorem of Tutte (see [1, Theorem 21.11]), stated below, is essential to prove that flower-snarks are 4-flow-critical.

**Theorem 2.1 (Tutte)** A cubic graph admits a 4-flow if and only if it admits a 3-edge-colouring.
A graph $G$ is $k$-colour-critical if it does not admit a $k$-vertex-colouring but every proper subgraph of $G$ admits a $k$-vertex-colouring. Dirac [4] defined the concept of critical graphs for the vertex-colouring problem and discovered several properties of such graphs. Likewise, in a previous paper [3] we defined the concept of (edge-) $k$-flow-critical graphs. A graph $G$ is $k$-flow-critical if it does not admit a $k$-flow but the graph $G/e$ obtained from $G$ by the contraction of an arbitrary edge $e$ does admit a $k$-flow. Moreover, as stated in [3, Theorem 3.1], a $k$-flow-critical graph can be equivalently defined as a graph $G$ that does not admit a $k$-flow but the graph $G - e$ obtained from $G$ by the removal of an arbitrary edge $e$ does admit a $k$-flow.

We will show in this section that every flower-snark is 4-flow-critical. It follows from the definition of snarks and Theorem 2.1, that those do not admit a 4-flow. Therefore, it suffices to show that if we remove any edge of a flower-snark, the resulting graph admits a 4-flow.

Let $e = (u, v)$ be any edge of a flower-snark $J_n$. As $J_n$ is cubic, $J_n - e$ is a subdivision of a cubic graph, with $u$ and $v$ being its only vertices of degree two. Let $H_e$ be the cubic graph obtained from $J_n - e$ by the contraction of two edges: one incident with $u$ and the other with $v$. We call $H_e$ the underlying cubic graph of $J_n - e$. It can be trivially shown that $H_e$ will have a 4-flow if and only if $J_n - e$ has a 4-flow. Therefore, it suffices to show that $H_e$ admit a 3-edge-colouring of its edges, for every edge $e$ of $J_n$.

Let $J_{a,b}$, for $1 \leq a < b \leq n - 1$, be the subgraph of $J_n$ induced by the edges in $E_i$ for $a \leq i \leq b$ and in $G[V_i]$ for $a < i < b$. We say $J_{a,b}$ has a clockwise rotated 3-colouring if it has a 3-edge-colouring such that the edges in $E_a$ receive a different colour each and those colours appear clockwise rotated in $E_b$, i.e., the pairs of edges $\{(z_a, z_{a+1}), (x_b, x_{b+1})\}$, $\{(x_a, x_{a+1}), (y_b, y_{b+1})\}$ and $\{(y_a, y_{a+1}), (z_b, z_{b+1})\}$ receive one different colour each. Figure 2(a) illustrates a clockwise rotated 3-colouring of $J_{a,a+1}$, for $1 \leq a \leq n - 2$. The colours assigned to the edges are represented by three different patterns of line drawings.

Similarly, we say $J_{a,b}$ has a counter-clockwise rotated 3-colouring if it has a 3-edge-colouring such that the edges in $E_a$ receive a different colour each and those colours appear counter-clockwise rotated in $E_b$, i.e., the pairs of edges $\{(z_a, z_{a+1}), (y_b, y_{b+1})\}$, $\{(x_a, x_{a+1}), (z_b, z_{b+1})\}$ and $\{(y_a, y_{a+1}), (x_b, x_{b+1})\}$ receive one different colour each. Figure 2(b) illustrates a counter-clockwise rotated 3-colouring of $J_{a,a+1}$, for $1 \leq a \leq n - 2$. The simple observation of the 3-colourings depicted on Figure 2 allow us to state the following lemma:

**Lemma 2.2** Every $J_{a,a+1}$ for $1 \leq a \leq n - 2$ has both a clockwise and a
counter-clockwise rotated 3-colouring.

Following the same idea, we say $J_{a,b}$ has a neutral 3-colouring if it has a 3-edge-colouring such that the edges in $E_a$ receive a different colour each and those colours appear replicated in $E_b$, i.e., the pairs of edges $\{(z_a, z_{a+1}), (z_b, z_{b+1})\}$, $\{(x_a, x_{a+1}), (x_b, x_{b+1})\}$ and $\{(y_a, y_{a+1}), (y_b, y_{b+1})\}$ receive one different colour each. In Figure 3 we present a neutral 3-colouring of $J_{a,a+2}$, for $1 \leq a \leq n-3$, and of $J_{a,a+3}$, for $1 \leq a \leq n-4$. The neutral 3-colouring of Figure 3(a) was obtained by combining a clockwise rotated 3-colouring of $J_{a,a+1}$ and a counter-clockwise rotated 3-colouring of $J_{a+1,a+2}$ whose colours coincide in $E_{a+1}$. Similarly, the neutral 3-colouring of Figure 3(b) was obtained by combining a sequence of three clockwise rotated 3-colourings of $J_{a,a+1}$, $J_{a+1,a+2}$ and $J_{a+2,a+3}$ whose colours coincide in both $E_{a+1}$ and $E_{a+2}$. These observations allow us to state the following corollary of Lemma 2.2.

**Corollary 2.3** Every $J_{a,a+2}$ for $1 \leq a \leq n-3$ and $J_{a,a+3}$ for $1 \leq a \leq n-4$ has a neutral 3-colouring.

The next result also follows from Lemma 2.2 and Corollary 2.3.

**Corollary 2.4** Graph $J_{1,n-1}$ has both a neutral and a clockwise rotated 3-colouring.

**Proof.** By Corollary 2.3, $J_{1,4}$ has a neutral 3-colouring and every $J_{a,a+2}$ for $4 \leq a \leq n-3$ also has a neutral 3-colouring. Adjust color names in such colourings so as to make them coincide in linking edges $E_a$ for $4 \leq a \leq n-3$. The union of those 3-colourings is a neutral 3-colouring of $J_{1,n-1}$.

Now, by Lemma 2.2, $J_{1,2}$ has a clockwise rotated 3-colouring and by Corollary 2.3, every $J_{a,a+2}$ for $2 \leq a \leq n-3$ also has a neutral 3-colouring. Adjust color names in such colourings so as to make them coincide in linking edges $E_a$ for $2 \leq a \leq n-3$. The union of those 3-colourings is a clockwise rotated
We will now show that for every edge \( e \) of \( J_n \) the underlying cubic graph \( H_e \) of \( J_n - e \) has a 3-edge-colouring. Recall that, due to the many symmetries of \( J_n \) there only four types of edges whose removal must be considered. We take \( e_1 = (z_1, z_n) \), \( e_2 = (w_1, x_1) \), \( e_3 = (z_1, w_1) \) and \( e_4 = (x_1, y_n) \) as representatives of internal, fork, spoke and external edges, respectively. Figure 4 illustrates the structure of the underlying cubic graphs \( H_{e_i} \), for \( i = 1, 2, 3, 4 \).

**Theorem 2.5** The underlying cubic graphs \( H_{e_i} \), for \( i = 1, 2, 3, 4 \) of any flower-snark \( J_n \) admit a 3-edge-colouring.

**Proof.** By Corollary 2.4, \( J_{1,n-1} \) has a neutral 3-colouring. Figures 4(a) and 4(b) show that such a 3-colouring can be extended to a 3-colouring of \( H_{e_1} \) and \( H_{e_2} \). Again, by Corollary 2.4, \( J_{1,n-1} \) has a clockwise rotated 3-colouring. Figures 4(c) and 4(d) show that such a 3-colouring can be extended to a 3-colouring of \( H_{e_3} \) and \( H_{e_4} \).

**Corollary 2.6** Every flower-snark is 4-flow-critical.

**References**


Fig. 4. 3-edge-colourings of underlying cubic graphs $H_{e_i}$, for $i = 1, 2, 3, 4$


